

On bifurcations in degenerate resonance zones

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Abstract

For Hamiltonian systems with $3/2$ degrees of freedom close to nonlinear integrable and for symplectic maps of the cylinder, bifurcations in degenerate resonance zones are discussed¹

1 Introduction

The study of orbit behavior in nonlinear nearly integrable Hamiltonian systems with $3/2$ degrees of freedom near the levels close to a resonance one was heavily advanced by L.P. Shil'nikov in the paper [1] (jointly with the author of this paper). These investigations were continued in numerous papers (references can be found in books [2]-[6]). In the papers mentioned resonances were classified as being passable, partially passable and non-passable. Also they can be non-degenerate or degenerate ones. Perturbations we consider can be Hamiltonian or non-Hamiltonian. Degenerate resonances occur in

¹The paper is based on the talk the author presented at the International Conference "Dynamics, Bifurcation, and Strange Attractors" dedicated to the memory of L.P.Shil'nikov (Nizhny Novgorod, Russia, July 1-5, 2013). Only the stuff devoted to degenerate resonances was included.

systems where the frequency of periodic orbits for an initial one-degree-of-freedom unperturbed Hamiltonian system is a non-monotone function of the value of Hamiltonian. The analysis of degenerate resonances was initiated in [1]. After that much work was done to analyze the degenerate resonances (see, for instance, [7]-[10], [6]).

For symplectic maps on the cylinder with a non-monotone rotation number, degenerate resonances were studied first in [11]. Here, we address mostly to bifurcations in degenerate resonance zones.

For systems under consideration, the motions in resonance zones (we call by this term some region near the resonance level of the unperturbed Hamiltonian) can be observed for a planar Poincaré map constructed by means of a computer (using, for instance, the package WinSet [12]). For sufficiently small perturbations, the pictures obtained in this way are in a good agreement with the analysis of averaged systems. In the case of Hamiltonian perturbations, the related Poincaré map preserves of course the area.

As for area preserving maps of the cylinder is concerned, the Chirikov map (or the standard map, in the other terminology) is the most popular. This is a map with a monotone rotation number. Also, maps were studied where the rotation function is nonlinear and non-monotone.

As the first example, maps with the quadratic rotation function $f(u) = p_1 u + p_2 x u^2$ (or $f(u) = p_1 + p_2 u^2$) with parameters p_1, p_2 were considered. Such the maps were called to be “standard maps with a non-monotone rotation” [11]. When studying, these maps were approximated by Hamiltonian flows and bifurcations of reconnecting separatrices were examined related with two scenarios: 1) the formation of “loops” and 2) the formation of “vortex pairs” [11]-[15]. Some related features were observed for the map itself as well. However, no answer was found so far to the following question: which resonances give rise to the formation of vortex pairs? For instance, in [13] vortex pairs arose for resonances with even number 2 but not for resonances with other integers.

When studying Hamiltonian systems with $3/2$ degrees of freedom close to nonlinear integrable, the second scenario has been numerically realized [10]. However, it was proved for sufficiently small perturbations that “vortex pairs” are absent there and only the first scenario for the reconnection of separatrices is possible. The existence of vortex pairs in [10] is explained by the fact that the perturbation was not “sufficiently small”. In this paper, we study in details the bifurcations in degenerate resonance zones, using the averaged system of the second approximation. This study answers the

question on the formation of vortex pairs.

2 Averaging

Let us consider a system

$$\begin{aligned}\dot{x} &= \frac{\partial H(x, y)}{\partial y} + \varepsilon g(x, y, \nu t) \\ \dot{y} &= -\frac{\partial H(x, y)}{\partial x} + \varepsilon f(x, y, \nu t),\end{aligned}\tag{1}$$

where $\varepsilon > 0$ is a small parameter, functions $H(x, y)$, $g(x, y, \nu t)$, $f(x, y, \nu t)$ are sufficiently smooth (or analytic) in variables x, y varying in some domain $D \subset R^2$ (or $D \subset S^1 \times R^1$) and continuous and $2\pi/\nu$ -periodic in t , ν is a parameter (the perturbation frequency). We assume the unperturbed system to be nonlinear and to possess a cell D_0 filled with periodic orbits. In D_0 the action-angle variables I, θ are introduced. As a result one obtains the system

$$\begin{aligned}\dot{I} &= \varepsilon[f(x, y, \varphi)x'_\theta - g(x, y, \varphi)y'_\theta] \equiv \varepsilon F(I, \theta, \varphi) \\ \dot{\theta} &= \omega(I) + \varepsilon[-f(x, y, \varphi)x'_I + g(x, y, \varphi)y'_I] \equiv \omega(I) + \varepsilon G(I, \theta, \varphi) \\ \dot{\varphi} &= \nu.\end{aligned}\tag{2}$$

Studying resonances play an essential role when analysing system (2). As is known, by a resonance in the system it is understood the presence of the following relation

$$\omega(I) = (q/p)\nu,\tag{3}$$

where p, q are co-prime integers. We denote the related value I in (3) as I_{pq} .

If the following conditions hold

$$\omega^{(k)}(I_0) = 0, \quad k = 1, 2, \dots, j-1; \quad \omega^{(j)}(I_0) \neq 0, \quad j > 1,\tag{4}$$

we call the level $I = I_0$ to be *degenerate* and the number j is its degeneracy order. If $\omega'(I_0) \neq 0$ (that is, $j = 1$), then the level $I = I_0$ is called to be *non-degenerate*.

By the resonance zone we mean henceforth a neighborhood

$$U_{\varepsilon^s} = \{(I, \theta) : I_{pq} - c\varepsilon^s < I < I_{pq} + c\varepsilon^s, 0 \leq \theta < 2\pi, c = \text{const} > 0, s = 1/(1+j)\}$$

of an individual resonance level $I = I_{pq}$. In this neighborhood, according to [6], the behavior of solutions to the initial system (2) is described (up to terms $O(\varepsilon^{1+s})$) by the autonomous system

$$\begin{aligned}\dot{u} &= \varepsilon^{1-s} A_0(v, I_{pq}) + \varepsilon P_0(v, I_{pq}) u \\ \dot{v} &= \varepsilon^{1-s} b_j u^j + \varepsilon (b_{j+1} u^{j+1} + Q_0(v, I_{pq})), \quad s = 1/(1+j),\end{aligned}\tag{5}$$

where $b_j = \omega^{(j)}(I_{pq})/j!$,

$$A_0(v, I_{pq}) = \frac{1}{2\pi p} \int_0^{2\pi p} F(I_{pq}, v + \frac{q}{p}\varphi, \varphi) d\varphi,\tag{6}$$

$$P_0(v, I_{pq}) = \frac{1}{2\pi p} \int_0^{2\pi p} [\partial F(I_{pq}, v + q\varphi/p, \varphi)/\partial I] d\varphi,\tag{7}$$

$$Q_0(v, I_{pq}) = \frac{1}{2\pi p} \int_0^{2\pi p} G(I_{pq}, v + q\varphi/p, \varphi) d\varphi.\tag{8}$$

Functions $A_0(v, I_{pq})$, $P_0(v, I_{pq})$, $Q_0(v, I_{pq})$ have a common period $2\pi/p$ in v . Let us write down functions $A_0(v, I_{pq})$, $P_0(v, I_{pq})$ in the form

$$\begin{aligned}A_0(v, I_{pq}) &= \widetilde{A}_0(v, I_{pq}) + B_0(I_{pq}), \\ P_0(v, I_{pq}) &= \widetilde{P}_0(v, I_{pq}) + B_1(I_{pq}),\end{aligned}\tag{9}$$

where $B_0(I_{pq})$ is the average value of $A_0(v, I_{pq})$ and $B_1(I_{pq})$ is the same for $P_0(v, I_{pq})$.

Now consider the equation

$$\widetilde{A}_0(v, I_{pq}) + B_0(I_{pq}) = 0,\tag{10}$$

whose roots define coordinates v of equilibria for the truncated system. Following [1] a resonance level $I = I_{pq}$ is called to be: i) passable, if the equation (10) has not real roots; ii) partially passable if (10) has only simple real roots and $B_0(I_{pq}) \neq 0$; iii) non-passable if $B_0(I_{pq}) = 0$.

System (5) is usually called to be the averaged system of the second approximation of the averaging method.

3 Bifurcations in degenerate resonance zones

Consider first the case of Hamiltonian perturbations. Then the initial system (1) is Hamiltonian and the following identities hold

$$P_0 + Q'_{0v} \equiv 0, \quad B_0 = B_1 = 0. \quad (11)$$

Hence, system (5) is also Hamiltonian. In this case, the first approximation system plays the main role in the analysis of phase curves of system (5)

$$\begin{aligned} \dot{u} &= \varepsilon^{1-s} A_0(v, I_{pq}) \\ \dot{v} &= \varepsilon^{1-s} b_j u^j \quad j \geq 2. \end{aligned} \quad (12)$$

The explicit calculation of $A(v, I_{pq})$ for main examples of nonlinear systems is relied on the problem of finding unperturbed solution $x(\theta, I_{pq}), y(\theta, I_{pq})$ which, in turn, is intimately connected with the problem of inversion for hyper-elliptic integrals.

System (12) has only degenerate equilibria. Therefore, along with system (12) its deformation is investigated

$$\begin{aligned} \dot{u} &= \varepsilon^{1-s} A(v, I_{p1}), \\ \dot{v} &= \varepsilon^{1-s} (b_j u^j + \sum_{k=1}^{j-1} b_k u^k), \quad s = 1/(j+1), \quad j \geq 2, \end{aligned} \quad (13)$$

where b_k are the deformation parameters. Systems in the form (13) were examined in [8, 9].

When the perturbation is Hamiltonian and harmonic (contains only one Fourier harmonics), then function $\tilde{A}(v, I_{pq})$ is also harmonic [6]. System (12) can be reduced to a form with $\tilde{A}(v, I_{pq}) = a_{pq} \sin pv$. We note that in this case functions $\tilde{P}_0(v, I_{pq}), Q_0(v, I_{pq})$ are also harmonic. Let us denote $\tilde{P}_0(v, I_{pq}) = c_{pq} \sin pv$. Then, due to (11), we have $Q_0(v, I_{pq}) = d_{pq} \cos pv$, and $c_{pq} = p d_{pq}$. For a harmonic perturbation it was shown that $q = 1$ [6].

Thus, for harmonic Hamiltonian perturbations, system (5) is reduced to the form

$$\begin{aligned} \dot{u} &= \varepsilon^{1-s} a_{p1} \sin pv + \varepsilon c_{p1} u \sin pv \\ \dot{v} &= \varepsilon^{1-s} b_j u^j + \varepsilon (b_{j+1} u^{j+1} + d_{p1} \cos pv), \quad s = 1/(1+j), \end{aligned} \quad (14)$$

In (14) we proceed with the slow time $\tau = \varepsilon^{1-s}t$. The Hamiltonian $\overline{H}(u, v)$ of system (14) takes the form

$$\overline{H}(u, v) = \frac{b_j u^{j+1}}{j+1} + \frac{a_{p1}}{p} \cos(pv) + \varepsilon^s \left(\frac{c_{p1}}{p} u \cos(pv) + \frac{b_{j+1}}{j+2} u^{j+2} \right), \quad j \geq 2. \quad (15)$$

Deformations of the vector field are described by a system with the Hamiltonian

$$\tilde{H}(u, v) = \overline{H}(u, v) + \sum_{k=1}^{j-1} \frac{b_k u^{k+1}}{k+1}. \quad (16)$$

We investigate further the averaged system

$$\frac{du}{d\tau} = -\frac{\partial \tilde{H}}{\partial v}, \quad \frac{dv}{d\tau} = \frac{\partial \tilde{H}}{\partial u} \quad (17)$$

for $j = 2$. In [10], an example was considered where a system in the form (17) with $j = 2$ was derived. Some phase portraits of system (17) were presented in [10], however the complete description was not found there.

3.1 Investigation of the averaged system. Bifurcations

Let us change $pv \rightarrow v$ in system (17) and denote $a = a_{p1}$, $b = b_2$, $\mu_1 = \varepsilon^{1/3} c_{p1}$, $\mu_2 = b_1$. Then, assuming $b_3 = 0$, we come to the Hamiltonian system

$$\begin{aligned} u' &= a \sin v + \mu_1 u \sin v \\ v' &= p(bu^2 + \mu_2 u) + \mu_1 \cos v \end{aligned} \quad (18)$$

with the Hamilton function $\tilde{H}(u, v) = p(\mu_2 u^2/2 + bu^3/3) + (a + \mu_1 u) \cos v$. Here the prime denotes the derivative in τ . To be definite, we set $a = 2$, $b = 1$. Let us forget for a minute on the smallness of the parameter ε .

System (18) has equilibria

$$\begin{aligned} O_1^\pm &((- \mu_2 \pm \sqrt{\mu_2^2 - 4\mu_1/p})/2, 0), \\ O_2^\pm &((- \mu_2 \pm \sqrt{\mu_2^2 + 4\mu_1/p})/2, \pi). \end{aligned}$$

System (18) can have more equilibria

$$O_3 \left(\frac{-2}{\mu_1}, \arccos \frac{2}{\mu_1^2} \left(p\mu_2 - \frac{2}{\mu_1} \right) \right),$$

$$O_4 \left(\frac{-2}{\mu_1}, \pi - \arccos \frac{2}{\mu_1^2} \left(p\mu_2 - \frac{2}{\mu_1} \right) \right),$$

if the following conditions are satisfied

$$-1 \leq \frac{2}{\mu_1^2} \left(p\mu_2 - \frac{2}{\mu_1} \right) \leq 1. \quad (19)$$

The characteristic equation has the form $\lambda^2 + \Delta = 0$ where Δ is the determinant of the Jacobi matrix, i.e., $\Delta = -\mu_1^2 \sin^2 v_0 - (2u_0 + p\mu_2)(2 + \mu_1 u_0) \cos v_0$, here u_0, v_0 are the coordinates of the equilibrium. Hence, system (18) can have simple equilibria of only two types: saddles and centers. If $\Delta < 0$ then the equilibrium is of the saddle type and if $\Delta > 0$, then it is a center.

Bifurcations of the equilibria O_1^\pm occur when parameters cross the parabola

$$p\mu_2^2 - 4\mu_1 = 0, \quad (20)$$

and for the equilibria O_2^\pm do on the parabola

$$p\mu_2^2 + 4\mu_1 = 0. \quad (21)$$

The equilibrium become double (degenerate) with $\lambda_{1,2} = 0$ at the bifurcation. Before the bifurcation two simple equilibria exist: a saddle and a center corresponding to the same value of v and different values of u . We shall call such bifurcations to be “vertical”. Before the bifurcation, meander type curves exist, in accordance to the terminology in [14].

Bifurcations of equilibria O_3, O_4 take place on the bifurcation curves

$$p\mu_2 = \frac{2}{\mu_1} \pm \frac{\mu_1^2}{2}. \quad (22)$$

Here at the moment of bifurcation, one has a triple saddle [16]. Unlike the previous case, the bifurcation here is “horizontal” (i.e. before the bifurcation, meander curves are absent, see Fig.2, VIII). The triple saddle breaks into two simple saddles and a center.

Besides the aforementioned local bifurcations, global bifurcations of reconnecting separatrices take place in system (18). They can may happen when the system has equilibria of the saddle type lying on different levels $u = u_1$ and $u = u_2$. Let v_1 and v_2 be corresponding coordinates v for these saddles. The equation the separatrix curves has the form $\tilde{H}(u, v) = h_1$, for the saddle

(u_1, v_1) with $h_1 = \tilde{H}(u_1, v_1)$ and the form $\tilde{H}(u, v) = h_2$ for the saddle (u_2, v_2) with $h_2 = \tilde{H}(u_2, v_2)$. The moment of reconnection bifurcation is defined by the equality

$$h_1(\mu_1, \mu_2) = h_2(\mu_1, \mu_2). \quad (23)$$

The related bifurcation curve separates domains I and II, IV and VII, V and VI on Fig.1 ($\mu_2 > 0$). All bifurcation curves (20)-(23) are shown on Fig.1. The bifurcation curves divide the plane of the parameters (μ_1, μ_2) into 20 domains (for bounded μ_1, μ_2). Due to the symmetry of these curves, it is sufficient to plot phase portraits from 12 domains in the upper half-plane $\mu_2 > 0$ (see Fig.1). Then the following bifurcation scenarios can occur.

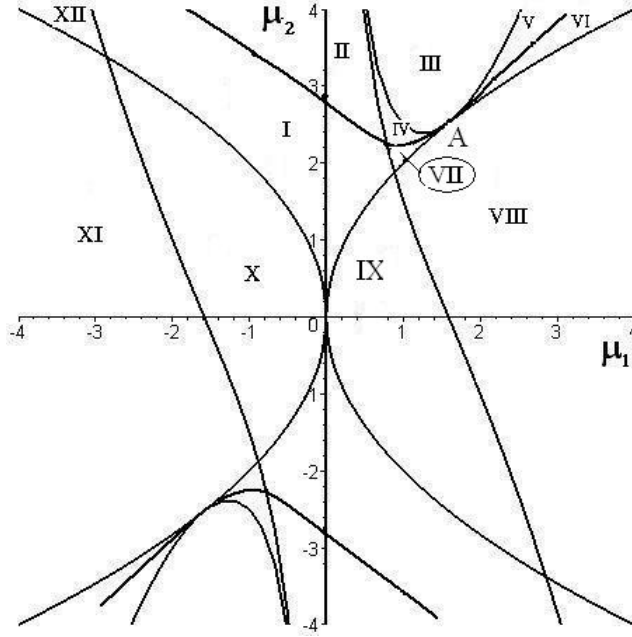


Figure 1: The domains in the parameter plane (μ_1, μ_2) with different topologies of phase portraits for system (18) at $a = 2$, $b = 1$ and $p = 1$.

- 1) The first scenario is related to a transition from domain I to domain II (bifurcations of “loops”).
- 2) The second scenario is related to formation (or annihilation) of “vortex pairs” (a transition from domain IX to domains VIII).

3) The third scenario of codimension 2 is related to a transition from domain III to domain VIII via the point A. At the bifurcation point, there is a degenerate saddle with 6 separatrices ([16], pp. 385-404).

Two types of reconnections are distinguished: 1) merging loops for saddles with $v = 0$ and $u = \pi$ (for example, a transition from domain I to domain II); 2) formation of a triangle from separatrices of three saddles with coordinates $|v| < \pi$ (for example, a transition from domain IV to domain VII).

Certainly, other scenarios may exist. As an example, we point out the closed path on the bifurcation diagram related to a transition from one domain to a neighboring one: $I \rightarrow II \rightarrow IV \rightarrow III \rightarrow V \rightarrow VI \rightarrow VIII \rightarrow IX \rightarrow X \rightarrow XI \rightarrow XII \rightarrow I$ and $I \rightarrow IX, I \rightarrow X; I \rightarrow VII \rightarrow VIII; IV \rightarrow VII$. Bifurcations that occur here can be understood using Fig.2.

The bifurcation diagram on Fig.1 depends on the parameters $a = a_{p1}$ and $b = d^2\omega(I_{p1})/2dI^2$. According to [6], in the analytic case, coefficient a_{p1} exponentially decays with growing p . Hence, for resonances with p large, the transition from domain I to domain VIII occurs for smaller values of $\mu_1 = \varepsilon^{1/3}c_{p1}$. The parameter c_{p1} also exponentially decays. Hence, trying to understand which scenario is realized, we conclude that this depends on interrelations of parameters a and μ_1 for the chosen parameters b and μ_2 . In any case, for μ_1 small enough only the first scenario is realized in system (18).

4 Area preserving maps of the cylinder

The averaged Hamiltonian system (18) is similar to systems derived when analysing symplectic maps of the cylinder with a non-monotone rotation [11]-[15]. In those papers, two bifurcation scenarios were also found: “loops” and “vortex pairs”. It is ascertained that the first scenario takes place for the principle resonance ($p = 1$), while the second scenario does for the sub-resonance of order 2 ($p = 2$). In [17], another map was considered being a combination of the Chirikov and the Fermi maps. For this map, vortex pairs exist also only for the sub-resonance of order 2.

In [11, 13], the standard map T with a non-monotone rotation depending on two parameters a, β is investigated

$$\begin{aligned} u_{n+1} &= u_n + a \sin v_n, \\ v_{n+1} &= v_n + u_{n+1} - \beta u_{n+1}^2, \quad n = 0, 1, 2, \dots \end{aligned} \tag{24}$$

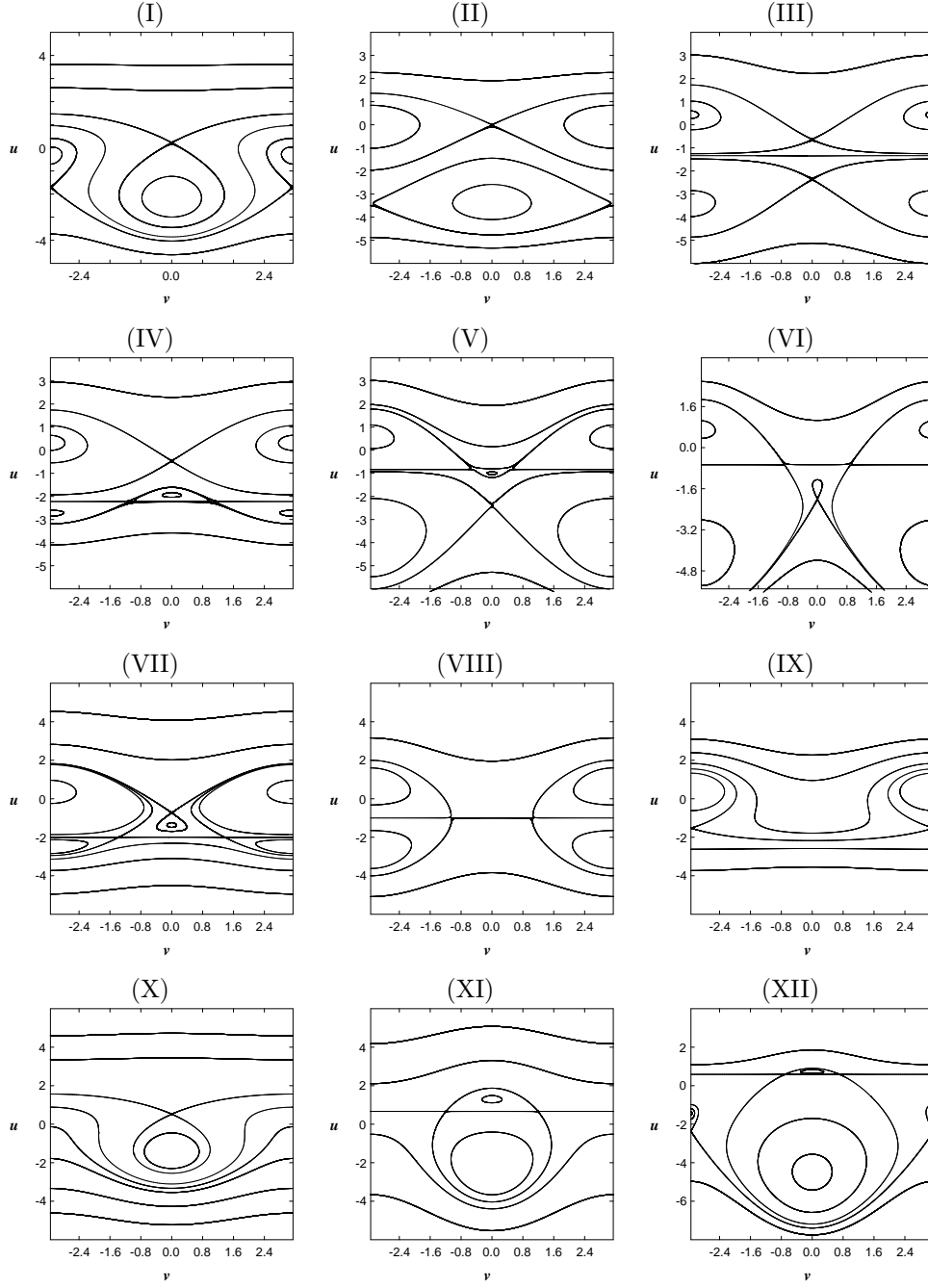


Figure 2: Phase portraits of system (18).

For this map, the approximating Hamiltonian flow is generated by the Hamilton function $H(u, v) = u^2/2 - \beta u^3/3 + a \cos v$, therefore only the first scenario of “loops” is realized. For the map T^2 , the approximating Hamiltonian flow is generated by the Hamilton function $H(u, v) = u^2/2 - \beta u^3/3 + (a^2/16)(1 - 2\beta u) \cos 2v$, and the second scenario of “vortex pairs” is realized. This bifurcation is related to a transition from domain III to domain VIII through the point A (see Fig.1). Both Hamiltonians are special cases of Hamiltonian (16).

Following [8], consider the map

$$\begin{aligned} u_{n+1} &= u_n + \alpha(a + \mu_1 u_{n+1}) \sin v_n, \\ v_{n+1} &= v_n + \alpha(pbu_{n+1}^2 + \mu_2 pu_{n+1} + \mu_1 \cos v_n), \quad n = 0, 1, 2, \dots, \end{aligned} \quad (25)$$

that is derived by means of the conservative Euler method to solve system (18). Here α is the approximation step when the passage from the differential system to the difference one, the error of the approximation is of the order α^2 . We recast (25) in the form²

$$\begin{aligned} u_{n+1} &= (u_n + \alpha a \sin v_n) / (1 - \alpha \mu_1 \sin v_n), \\ v_{n+1} &= v_n + \alpha(pbu_{n+1}^2 + \mu_2 pu_{n+1} + \mu_1 \cos v_n), \quad n = 0, 1, 2, \dots \end{aligned} \quad (26)$$

For small α , invariant curves of map (26) are close to orbits for system (18). Hence, the first and the second bifurcation scenaria are realized. However, it is worth noting two discrepancies: 1) if system (18) has a separatrix going from a saddle to a saddle then the separatrices of map (26) are split generating a Poincaré homoclinic structure (Fig. 3a); 2) the map has a boundary invariant curve embracing the cylinder (see [18] in this connection). Above this curve (on the upper half-cylinder) or below it (on the lower half-cylinder), orbits are wandering [8] (see Fig. 3b).

5 Conclusion

The analysis of systems with 3/2 degrees of freedom being close to two-dimensional integrable leads to the analysis of two-dimensional systems (5) on the cylinder. The latter defines an approximating flow for the Poincaré map induced by the initial system. In the case under consideration, the constructed map (26) defines the main features of the map.

²Similar maps were considered in [14].

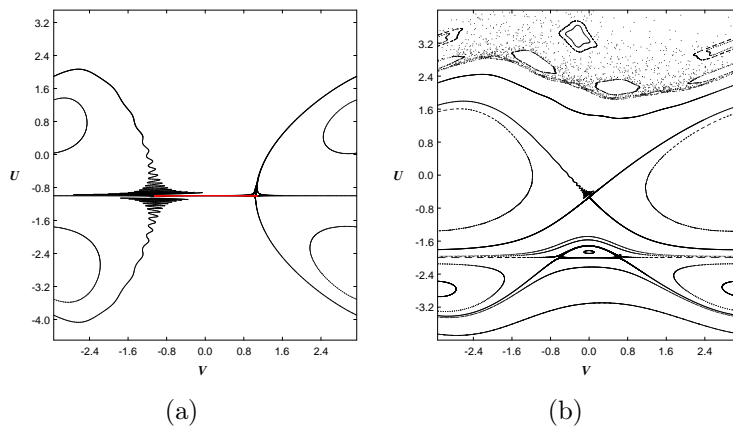


Figure 3: The trajectories of map (26) for $\alpha = 0.17$, $a = 2$, $b = 1$ corresponding to Fig.2 (VIII, IV).

For Hamiltonian systems studied above, phase portraits of system (18) provide all possible phase portraits constructed for Hamiltonian flows approximating area preserving maps of the cylinder with a non-monotone rotation [11]-[15]. System (18) is a special case of system (5).

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